

*The Thermal Stresses in Solid and in Hollow Circular Cylinders
concentrically Heated.*

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1. The large differences of temperature which exist between the inside and outside of the heating or firing tunnels used in pottery and other works, and between the inner and outer portions of the concrete or masonry pillars supporting the floors of a building in which a fire occurs, make the thermal stresses approach the breaking strength of the materials and they are crushed or torn asunder. The same may be said to a less degree of chimney stacks, and of the walls of the cylinders of internal combustion engines.

In order to prevent this, it is necessary to design the structure so as to withstand the stresses which will be produced by the differences of temperature to which they will be subjected. The calculation of these stresses has only been carried out for a hollow cylinder of constant elasticity and heat conductivity, with the temperature throughout it following the steady logarithmic law and the material expanding according to the linear law.*

It is proposed to treat the problem in a more general way in the present communication, and to show that for any temperature distribution, with any law of expansion, the stresses may be found by a simple graphical method so long as the variation of the elastic constants with temperature may be neglected.

Determination of the Stresses.

2. Let a hollow circular cylinder, of internal radius r_1 and external radius r_2 , and considerable length, originally at constant temperature, and not subjected to any pressure, have its temperature raised so that the temperature of each point of a cylindrical surface of radius r within it is raised to θ , where θ is a function of r . Let the cylinder then be subjected to a uniform pressure, R_1 , at its inner, a uniform pressure, R_2 , at its outer curved surface, and a uniform pressure, X , at each of its end surfaces. To find the mechanical and thermal stresses at each point of the material of the cylinder.

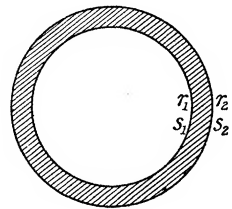


FIG. 1.—Radii, r_1 and r_2 ; sectional areas, s_1 and s_2 .

In the first instance we note that a uniform rise of temperature would produce a uniform expansion of the cylinder, and no stresses would result

* See R. Lorenz, 'Zeitsch. Verein Deutsch. Ing.', vol. 51, p. 743 (1907).

from such an expansion. We therefore divide the final temperature, θ , into two parts: first, a uniform temperature, $\bar{\theta}$, to which the cylinder would have to be heated to give the same expansion of the whole shell as that produced by the actual temperature, θ ; second, the excess, θ' , of the actual temperature over this uniform temperature.

Thus
$$\theta = \bar{\theta} + \theta'.$$

Whatever the character of the law of expansion of the material of the shell with rise of temperature, we may write $v_\theta = v_0(1 + \alpha_\theta)$ where v_0 is the volume of an element at 0° C. and v_θ at θ° C., and α_θ is the increase of volume of 1 c.c. when raised from 0° C. to θ° C. Hence if $\bar{\alpha}$ is the mean expansion for the whole shell and α_θ' the excess at a point where the temperature is θ over the mean expansion, we have $\alpha_\theta = \bar{\alpha} + \alpha_\theta'$.

Next we notice that, although the pressures on the end surfaces are uniform the pressures in planes parallel to them cease to be uniform, owing to the variation of θ' with r . In planes at distances from the end surfaces greater than the thickness of the cylinder these pressures become independent of the positions of the planes.

In the middle length of the cylinder points initially in a plane perpendicular to the axis of the cylinder at x are still in a plane perpendicular to the axis and at $x+x'$ after the expansion, and the initial distance, dx apart of two such planes becomes $(1 + (dx'/dx))dx$ after expansion, and is independent of r . The same statement would hold for the whole length of the cylinder, provided the end surfaces were kept plane by pressures or tractions applied in a suitable way over them.

Under these circumstances the only stresses acting on an elementary portion of the material bounded by two planes perpendicular to the axis at distance x and $x+dx$ from some fixed plane, by two cylindrical surfaces of radii r and $r+dr$, and two planes through the axis making angles ϕ and $\phi+d\phi$ with some fixed axial plane, are the equal axial pressures, X , acting on the surfaces perpendicular to the axis, the radial pressures, R and $R+dR$, on the

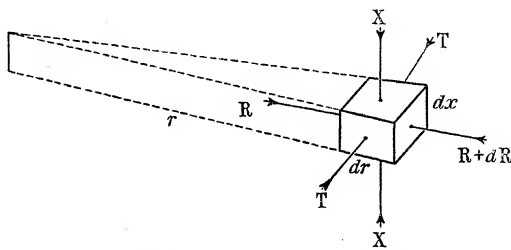


FIG. 2.—The pressures— R , radial; T , tangential; and X , axial, at the faces of an element.

inner and outer cylindrical surfaces respectively, and the equal tangential pressures, T , acting on the axial surfaces at right angles to those surfaces (fig. 2). If the displacement outwards of a point originally r from the axis is r' , the stress-strain equations for the elementary portion of the material are

$$\frac{1}{\epsilon} \left\{ -R + \sigma (T + X) \right\} = \frac{dr'}{dr} - \frac{1}{3} \alpha_{\theta}', \quad (2.1)$$

$$\frac{1}{\epsilon} \left\{ -T + \sigma (R + X) \right\} = \frac{r'}{r} - \frac{1}{3} \alpha_{\theta}', \quad (2.2)$$

$$\frac{1}{\epsilon} \left\{ -X + \sigma (R + T) \right\} = \frac{dx'}{dx} - \frac{1}{3} \alpha_{\theta}', \quad (2.3)$$

where ϵ is Young's modulus, σ Poisson's ratio for the material, and dx'/dx is independent of r and x and may be written α_x .

The stress equilibrium equation is

$$T = R + r \frac{dR}{dr}, \quad (2.4)$$

Multiplying (2.3) through by σ and adding to (2.2) we get

$$\frac{1}{\epsilon} \left\{ -T + \sigma R + \sigma^2 (R + T) \right\} = \frac{r'}{r} + \sigma \alpha_x - \frac{1 + \sigma}{3} \alpha_{\theta}'. \quad (2.5)$$

Subtracting (2.2) from (2.1) we get

$$\frac{1 + \sigma}{\epsilon} (T - R) = \frac{dr'}{dr} - \frac{r'}{r}.$$

Substituting for $T - R$ from (2.4) this gives

$$\frac{1 + \sigma}{\epsilon} \frac{dR}{dr} = \frac{d}{dr} \left(\frac{r'}{r} \right) \quad (2.6)$$

and

$$\frac{1 + \sigma}{\epsilon} R = \frac{r'}{r} + \text{a constant.}$$

Differentiating (2.5) with respect to r , assuming that the variations of ϵ and σ with the radius are negligible and subtracting the result from (2.6) we get

$$\frac{1 - \sigma^2}{\epsilon} \frac{d}{dr} (T + R) = \frac{1 + \sigma}{3} \frac{d\alpha_{\theta}'}{dr}.$$

Integrating this gives us

$$T + R = \frac{1}{3} \frac{\epsilon}{1 - \sigma} \alpha_{\theta}' + A', \quad (2.7)$$

where A' is a constant of integration.

Substituting for T from (2.4) we get

$$2R + r \frac{dR}{dr} = \frac{1}{3} \frac{\epsilon}{1 - \sigma} \alpha_{\theta}' + A'.$$

Multiplying through by r and integrating we have

$$r^2 R = B' + A' \frac{r^2}{2} + \frac{1}{3} \frac{\epsilon}{1-\sigma} \int \alpha_{\theta}' \cdot r dr.$$

The form of this equation suggests the expression of the stresses in terms of $\frac{1}{3}$ the area of the circle of radius r instead of r . The equation becomes

$$Rs = As + B + \frac{1}{6} \frac{\epsilon}{1-\sigma} \int \alpha_{\theta}' \cdot ds, \quad (2.8)$$

where $A = \frac{\pi A'}{2}$ and $B = \pi B'$.

Hence R_1 and R_2 , the pressures on the inside and outside surfaces respectively, satisfy the equations

$$R_1 s_1 = A s_1 + B + \frac{1}{6} \frac{\epsilon}{1-\sigma} \int_{s_1}^{s_1} \alpha_{\theta}' ds,$$

$$R_2 s_2 = A s_2 + B + \frac{1}{6} \frac{\epsilon}{1-\sigma} \int_{s_1}^{s_2} \alpha_{\theta}' ds.$$

Hence $Rs - R_1 s_1 = A(s - s_1) + \frac{1}{6} \frac{\epsilon}{1-\sigma} \int_{s_1}^s \alpha_{\theta}' ds$

and $R_2 s_2 - R_1 s_1 = A(s_2 - s_1)$ since $\int_{s_1}^{s_2} \alpha_{\theta}' ds = 0$.

On elimination of A these give

$$R = \frac{R_2 s_2 - R_1 s_1}{s_2 - s_1} + \frac{(R_1 - R_2) s_1 s_2}{s_2 - s_1} \frac{1}{s} + \frac{1}{6} \frac{\epsilon}{1-\sigma} \frac{E'}{s}, \quad (2.9)$$

where

$$E' = \int_{s_1}^s \alpha_{\theta}' \cdot ds = \int_{s_1}^s (\alpha_{\theta} - \bar{\alpha}) ds = E_{\theta} - \bar{\alpha}(s - s_1) = E_{\theta} - E_2 \frac{s - s_1}{s_2 - s_1},$$

E_2 being the value of E_{θ} at the outer surface.

Or if u be written for $1/s$ throughout the coefficients

$$R = \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + \frac{R_1 - R_2}{u_1 - u_2} \frac{1}{s} + \frac{1}{6} \frac{\epsilon}{1-\sigma} \frac{E'}{s}. \quad (2.10)$$

Equation (2.4) becomes $T = R + 2s \frac{dR}{ds}$ hence

$$T = \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} - \frac{R_1 - R_2}{u_1 - u_2} \frac{1}{s} + \frac{1}{6} \frac{\epsilon}{1-\sigma} \left(2\alpha_{\theta}' - \frac{E'}{s} \right). \quad (2.11)$$

On substituting for R and T in equation (2.3) we get

$$X = -\epsilon a_x + 2\sigma \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + \frac{1}{6} \frac{\epsilon}{1-\sigma} 2\alpha_{\theta}'. \quad (2.12)$$

On substituting for R, T, and X in equation (2.2) we get

$$\frac{r'}{r} = \frac{1}{2} \cdot \frac{s'}{s} = \frac{1+\sigma}{\epsilon} \left\{ -(1-2\sigma) \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + \frac{R_1 - R_2}{u_1 - u_2} \cdot \frac{1}{s} \right\} - \sigma a_x + \frac{1}{6} \frac{1+\sigma}{1-\sigma} \frac{E'}{s}. \quad (2.13)$$

The fractional increase of length a_x in the axial direction and the part X_1 of X which is independent of r and x may be given any arbitrary values so long as their sum $X_1 + \epsilon a_x$ is kept equal to $2\sigma \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2}$.

Thus if the increase of length a_x is taken zero $X_1 = 2\sigma \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2}$, while if X_1 is taken zero $a_x = 2\sigma \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2}$.

Whatever value is assigned to a_x it only affects the expressions for the mechanical portions of the stresses and displacements, the terms for the thermal stresses and displacements are unaffected. They depend on E' and $dE'/ds = \alpha_\theta'$ only.

The mean pressure $\frac{1}{3}(R+T+X)$ at the point r is given by

$$\frac{1}{3}(R+T+X) = \frac{2}{3}(1+\sigma) \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} - \frac{\epsilon}{3} a_x + \frac{1}{6} \frac{\epsilon}{1-\sigma} \frac{4\alpha_\theta'}{3}, \quad (2.14)$$

the shear stress in the plane T, X is given by

$$\frac{1}{3}(X-T) = -\frac{1-2\sigma}{3} \cdot \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + \frac{1}{3} \frac{R_1 - R_2}{u_1 - u_2} \cdot \frac{1}{s} - \frac{\epsilon}{3} a_x + \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{1}{3} \frac{E'}{s}, \quad (2.15)$$

and in the plane R, T by

$$\frac{1}{3}(R-T) = \frac{2}{3} \frac{R_1 - R_2}{u_1 - u_2} \cdot \frac{1}{s} + \frac{1}{6} \frac{\epsilon}{1-\sigma} \left(-\frac{2}{3} \alpha_\theta' + \frac{2}{3} \frac{E'}{s} \right), \quad (2.16)$$

and in the plane R, X by

$$\begin{aligned} \frac{1}{3}(R-X) = \frac{1-2\sigma}{3} \cdot \frac{R_2 u_1 - R_1 u_2}{u_1 - u_2} + \frac{1}{3} \frac{R_1 - R_2}{u_1 - u_2} \cdot \frac{1}{s} + \frac{\epsilon}{3} a_x \\ + \frac{1}{6} \frac{\epsilon}{1-\sigma} \left(\frac{1}{3} \frac{E'}{s} - \frac{2}{3} \alpha_\theta' \right). \end{aligned} \quad (2.17)$$

The terms involving the internal and external pressures are identical with those given by Lamé in his *Leçons* and quoted by Love.

If the cylinder is solid, the constant of integration B in equation (2.8) becomes zero. On making this change in the calculations giving the stresses it is found that the term $\frac{R_2 u_1 - R_1 u_2}{u_1 - u_2}$ in the expression for each stress reduces to R_2 —the pressure on the outside surface—the term involving $R_1 - R_2$

* Love, 'Theory of Elasticity,' 2nd edit., p. 141 (1906).

disappears, that involving α_z remains, while the terms giving the thermal part of the stresses remain unchanged in form, but E' is now $\int_0^s \alpha_\theta' \cdot ds$ and $\frac{E'}{s}$ at $s = 0$ becomes $\left(\frac{dE'}{ds}\right)_0$ that is α_θ' at the centre, so that the thermal radial pressure does not become zero at the centre, but the shear stress in the R, T plane does.

Graphical Constructions for the Stresses and Displacements.

3. The foregoing expressions for the stresses and displacements lead to the following methods of representing and determining both graphically.

A. The Stresses Due to the Pressures on the Inner and Outer Surfaces.

Set off along the axis of abscissæ, fig. 3, the reciprocals $OU_1 = u_1$ and $OU_2 = u_2$ of the areas s_1 and s_2 of the circles bounding the normal section of the cylinder. At U_2 erect the ordinate U_2R_2 , equal on some convenient scale to the pressure R_2 on the outside surface. Join the end R_2 of this ordinate to the point U_1 , and produce to cut the axis of pressure in S_0 . Through S_0 draw a straight line parallel to the axis of abscissæ, and let it cut the ordinates through U_2 and U_1 in S_2 and S_1 . Set off above S_1 a length S_1R_1 , equal to R_1 the pressure on the inner surface, join the end of this ordinate to S_2 , and produce to cut the axis of pressure in S_0' .

Then OS is $R_2 \cdot \frac{u_1}{u_1 - u_2}$, S_0S_0' is $R_1 \cdot \frac{u_2}{u_1 - u_2}$, hence $OS_0' = \frac{R_2u_1 - R_1u_2}{u_1 - u_2}$, that is the constant term in the expressions for the stresses.

To get the term involving $\frac{R_1 - R_2}{u_1 - u_2} u$, erect at U_1 , fig. 4, the ordinate $U_1A = R_1 - R_2$, join the end of it to the point U_2 and draw through O a straight line OB parallel to this join. The ordinate US at any point U is then $= \frac{R_1 - R_2}{u_1 - u_2} \cdot u$.

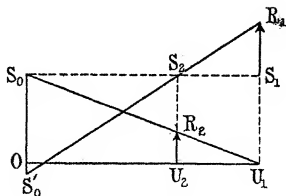


FIG. 3.

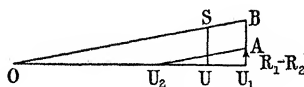


FIG. 4.

FIG. 3.— $OU_2 = u_2$, $OU_1 = u_1$, OS_0' the mean pressure (tension) determined by U_2R_2 , $U_1R_2S_0$, $S_0S_2S_1$, S_1R_1 and $R_1S_2S_0$.

FIG. 4.—The variable stress US determined from $U_1A = R_1 - R_2$, U_2A and OSB .

The mechanical stresses are then given as follows :

$$R = OS_0' + US, \quad (3.1)$$

$$T = OS_0' - US, \quad (3.2)$$

$$X = -\epsilon a_x + 2\sigma \cdot OS_0', \quad (3.3)$$

$$\frac{r'}{r} = \frac{1+\sigma}{\epsilon} \{ -(1-2\sigma) OS_0' + US \} - \sigma a_x, \quad (3.4)$$

the mean pressure

$$\frac{1}{3} (R+T+X) = \frac{2}{3} (1+\sigma) OS_0' - \frac{\epsilon}{3} a_x, \quad (3.5)$$

the shear stresses

$$\frac{1}{3} (X-T) = -\frac{1-2\sigma}{3} OS_0' + \frac{1}{3} US - \frac{\epsilon}{3} a_x, \quad (3.6)$$

$$\frac{1}{3} (R-T) = \frac{2}{3} US, \quad (3.7)$$

and
$$\frac{1}{3} (R-X) = -\frac{1-2\sigma}{3} OS_0' - \frac{1}{3} US + \frac{\epsilon}{3} a_x. \quad (3.8)$$

B. Stresses Due to the Excess of the Dilatation at a Point over the Mean Dilatation.

Set off as abscissæ OA_1 OA_2 the areas s_1 and s_2 of the inner and outer boundaries of the section of the cylinder. As ordinates take the temperature θ of each narrow ring of internal area s , and draw the curve of temperature distribution throughout the section, fig. 5.

For each temperature estimate from tables of dilatation of the material, the dilatation α_θ of 1 c.c. of each part of the section, and with α_θ as ordinates, draw the curve of dilatation per cubic centimetre D to correspond to the temperature curve.

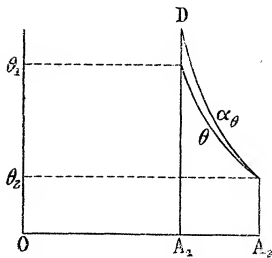


FIG. 5.

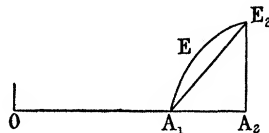


FIG. 6.

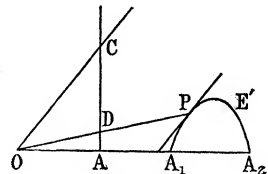


FIG. 7.

FIG. 5.— $OA_1 = s_1$, $OA_2 = s_2$, θ the curve of temperature, and α_θ that of dilatation.

FIG. 6.—The expansion curve E , $OA_1 = s_1$, $OA_2 = s_2$.

FIG. 7.— $A_1E'A_2$, the excess expansion curve ; OC parallel to the tangent at P ;
ADC a convenient ordinate.

Find the total expansion E of the shells within the radius r , if they had expanded independently of each other, that is, find the area enclosed between the axis of area, the curve D , the ordinate through A_1 and that through s for different values of s up to s_2 , and plot the values obtained as ordinates against the areas as abscissæ. Join the end of the curve at E_2 with its starting point at A_1 , fig. 6.

The excess of the ordinate of the curve at any point over the ordinate of the join is the ordinate of the excess expansion curve E' required, fig. 7, the scale of ordinates being for convenience increased.

Join P a point on the curve to the origin O . Through O draw OB parallel to the tangent to the E' curve at P and at A , corresponding to area s , a convenient multiple of 10, erect a perpendicular to the area axis cutting OB in C and OP in D_1 .

Then the thermal stresses are given as follows :

$$R = \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot AD/OA, \quad (3.1')$$

$$T = \frac{1}{6} \frac{\epsilon}{1-\sigma} (2AC - AD)/OA, \quad (3.2')$$

$$X = \frac{1}{6} \frac{\epsilon}{1-\sigma} 2 \frac{AC}{OA}, \quad (3.3')$$

$$\frac{r'}{r} = \frac{1}{6} \frac{1+\sigma}{1-\sigma} \frac{AD}{OA}, \quad (3.4')$$

the mean pressure

$$\frac{1}{3} (R+T+X) = \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{4}{3} \cdot \frac{AC}{OA}, \quad (3.5')$$

the shear stresses

$$\frac{1}{3} (X-T) = \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{1}{3} \frac{AD}{OA}, \quad \frac{1}{3} (R-T) = \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{2}{3} \frac{AD-AC}{OA}, \quad (3.6' \text{ \& } 7')$$

and

$$\frac{1}{3} (R-X) = \frac{1}{6} \frac{\epsilon}{1-\sigma} \frac{1}{3} \cdot \frac{AD-2AC}{OA}. \quad (3.8')$$

It will be noted that for the thermal stresses $R+T=X$ at any point, and since at the surfaces $R=0$, $T=X$ at them both.

NUMERICAL EVALUATION OF STRESSES IN PRACTICAL CASES.

4. In applying the foregoing theory to practical problems it seems desirable to select in the first instance cases of a comparatively simple type in which the stresses due to mechanical causes are small. Two such cases are (1) that of a long pottery kiln, a section of which is kept at the firing temperature

either permanently or alternately with its neighbours, and (2) that of a circular pillar of concrete or of masonry supporting the ceiling of a warehouse in which a fire occurs.

Case (1) a long Pottery Kiln or Furnace.

5. The distribution of temperature throughout a cylinder concentrically heated and of length sufficient to make the effect of the ends negligible is determined by the equation

$$c\rho \frac{\partial \theta}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(rk \frac{\partial \theta}{\partial r} \right). \quad (5.1)$$

If, in the first instance, we take the density ρ , the specific heat c and the heat conductivity k as constant, and the temperature distribution throughout the wall of the furnace as steady, θ_1 being the temperature of the inner surface at r_1 , θ_2 that of the outer surface at r_2 , we have

$$\frac{\theta - \theta_2}{\theta_1 - \theta_2} = \frac{\log r_2 - \log r}{\log r_2 - \log r_1} = \frac{\log s_2 - \log s}{\log s_2 - \log s_1}. \quad (5.2)$$

For simplicity we may take the dilatation α_θ proportional to the rise of temperature, that is $\alpha_\theta = \alpha\theta$. The expansion E of the rings between s_1 and s , if each expands independently of its neighbours, is given by $E = \int_{s_1}^s \alpha(\theta - \theta_2) ds$, and may be found either by graphic integration of the curve of temperature or directly from the equation

$$E = \alpha(\theta_1 - \theta_2) \frac{1}{\log s_2 - \log s_1} \{s(\log s_2 - \log s) - s_1(\log s_2 - \log s_1) + s - s_1\}, \quad (5.3)$$

and plotted as in fig. 8, where the temperature and the coefficient of $\alpha(\theta_1 - \theta_2)$ in E are shown for a cylinder when $s_1 = 0.75s_2$. In that case the temperature curve is nearly linear.

The total expansion of the area of section of the kiln wall is given by the ordinate at A_2 of the E curve, fig. 8, and has the value

$$E_2 = \frac{\alpha(\theta_1 - \theta_2)}{\log s_2/s_1} \left\{ -s_1 \log \frac{s_2}{s_1} + s_2 - s_1 \right\}.$$

The excess expansion $E' = E - E_2 \frac{s - s_1}{s_2 - s_1}$ is represented by the excess of the ordinate of the E curve over that of the join of the two ends of that curve in fig. 8 and is given by

$$E' = \alpha(\theta_1 - \theta_2) \left\{ s \frac{\log s_2 - \log s}{\log s_2 - \log s_1} - s_1 \frac{s_2 - s}{s_2 - s_1} \right\}. \quad (5.4)$$

The excess of the dilatation at any point over the mean dilatation for the whole wall is represented by the slope of the E' curve, and is given by

$$\alpha\theta' = \frac{dE'}{ds} = \alpha(\theta_1 - \theta_2) \left\{ \frac{\log s_2 - \log s - 1}{\log s_2 - \log s_1} + \frac{s_1}{s_2 - s_1} \right\}.$$

On drawing the lines OP, OB and the vertical ADC, corresponding to fig. 7,

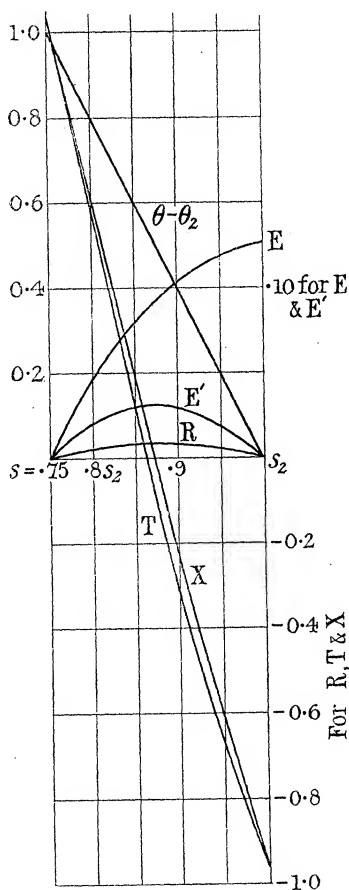


FIG. 8.—Temperature, expansion and stress curves for a hollow cylinder heated within.

or on substituting the values found above in the expressions for the thermal stresses and displacements, we get

$$R = \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{AD}{OA} = \frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha(\theta_1 - \theta_2) \left\{ \frac{\log s_2 - \log s - \frac{s_2}{s} - 1}{\log s_2 - \log s_1 - \frac{s_2}{s_1} - 1} \right\}. \quad (5.5)$$

Thus the radial pressure is zero at the inner surface, rises to a maximum

$$\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \alpha(\theta_1 - \theta_2) \left\{ \frac{1}{(\frac{s_2}{s_1} - 1) - 1} + \frac{s_2}{s_1} - 1 - \frac{1}{\log s_2 - \log s_1} \right\}$$

at the section $s = s_1 s_2 \frac{\log s_2 - \log s_1}{s_2 - s_1}$, or $s = s_2 \frac{\log s_2/s_1}{(s_2/s_1) - 1}$, and becomes zero at the outer surface. If the outer temperature θ_2 is higher than the inner θ_1 these pressures become tensions.

In the same way the circumferential pressure T is given by

$$T = \frac{1}{6} \frac{\epsilon}{1-\sigma} \frac{2AC-AD}{OA} = \frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha (\theta_1 - \theta_2) \left\{ \frac{s_2/s + 1}{s_2/s_1 - 1} - \frac{2 - (\log s_2 - \log s)}{\log s_2 - \log s_1} \right\}. \quad (5.6)$$

Thus the circumferential pressure is

$$\frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha (\theta_1 - \theta_2) \left\{ \frac{2s_2}{s_2 - s_1} - \frac{2}{\log s_2 - \log s_1} \right\}$$

at the inner surface r_1 , becomes zero slightly within the ring of sectional area $(s_1 + s_2)/2$, and is $-\frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha (\theta_1 - \theta_2) \left\{ -\frac{2s_1}{s_2 - s_1} + \frac{2}{\log s_2 - \log s_1} \right\}$, that is a tension at the outer surface r_2 .*

The longitudinal pressure X is given by

$$X = \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot 2 \frac{AD}{OA} = \frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha (\theta_1 - \theta_2) 2 \left\{ \frac{\log s_2/s - 1}{\log s_2/s_1} + \frac{s_1}{s_2 - s_1} \right\}. \quad (5.7)$$

It changes from $\frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha (\theta_1 - \theta_2) 2 \left\{ \frac{s_2}{s_2 - s_1} - \frac{1}{\log s_2/s_1} \right\}$ at the inner surface to zero at the surface $\frac{s}{s_2} e^{\left(\frac{s_2}{s_1}\right)^{s_1/(s_2-s_1)}}$ and becomes

$$-\frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha (\theta_1 - \theta_2) 2 \left\{ \frac{1}{\log s_2/s_1} - \frac{s_1}{s_2 - s_1} \right\}$$

at the outer surface.

The thermal displacement other than that due to the mean rise of temperature is given by

$$\frac{r'}{r} = \frac{1}{2} \frac{s'}{s} = \frac{1}{6} \frac{1+\sigma}{1-\sigma} \frac{AD}{OA} = \frac{1}{6} \frac{1+\sigma}{1-\sigma} \alpha (\theta_1 - \theta_2) \left\{ \frac{\log s_2/s}{\log s_2/s_1} - \frac{s_2/s - 1}{s_2/s_1 - 1} \right\}. \quad (5.8)$$

Thus the excess thermal displacement is zero at the two surfaces and has its maximum value $\frac{1}{6} \frac{1+\sigma}{1-\sigma} \alpha (\theta_1 - \theta_2) \frac{1}{\log s_2/s_1} \left\{ \log \left(\frac{s_2/s_1 - 1}{\log s_2/s_1} \right) - 1 \right\}$ at the surface $\frac{s}{s_2} = \frac{\log s_2/s_1}{s_2/s_1 - 1}$.

To this displacement must be added that due to the mean rise of temperature which is given by $\frac{\bar{r}'_0}{r} = \frac{1}{2} \frac{\bar{s}'_0}{s} = \frac{1}{3} \alpha (\theta_1 - \theta_2) \left\{ \frac{s_2 - s_1}{\log s_2/s_1} - s_1 \right\}$.

* The values of the tangential and axial stresses at the two surfaces are identical with those given by R. Lorenz, *loc. cit.*

The coefficients of $\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \alpha (\theta_1 - \theta_2)$ in the expressions for the stresses are shown in fig. 8 as functions of s for the case $s_1 = 0.75s_2$. If $\theta_1 - \theta_2 = 1000^\circ$ C., the coefficient of cubical expansion of the material $\alpha = 12 \times 10^{-6}$, Young's modulus $\epsilon = 10^{11}$ and Poisson's ratio $\sigma = \frac{1}{3}$, the value of $\frac{1}{6} \frac{\epsilon}{1-\sigma} \alpha (\theta_1 - \theta_2)$ is 3×10^8 dynes per sq. cm. The tensile strength of fire brick is of the order 2×10^7 dynes per square centimetre (300 lbs. per square inch). The tensions T and X in the outer portion of the wall are, therefore, capable of producing cracks in the material which will be radial, and, since at the outer surface T and X are equal, will extend in the longitudinal as often as in the transverse direction.

6. If the temperature throughout the section is not steady but variable, the number of possible distributions is infinite, but of these cases two are worth investigating as of practical value (A) when the kiln or furnace initially at θ_2 has its inner surface raised at a given instant to θ_1 , and (B) when a kiln or furnace has been in operation with its inner surface at θ_1 and its outer at θ_2 for a considerable time, and at a given time has the temperature of its inner surface reduced to θ_2 .

A. The kiln is brought into service.

For the first few minutes the only change of temperature is that close to the inner surface. The mean temperature is scarcely changed from θ_2 , the E' curve has very small ordinates but θ' has approximately the value $\theta_1 - \theta_2$ close to the inner surface.

Hence at this stage, close to the inner surface, R the radial pressure and r'/r the radial displacement are very small, but T the tangential and X the longitudinal pressures are both very nearly double what they finally become in the steady state, that is near the inner surface they both have initially the value

$$\frac{1}{3} \frac{\epsilon \alpha}{1-\sigma} (\theta_1 - \theta_2),$$

and decrease to half that value as the temperature distribution approaches that of the steady state.

B. The kiln is put out of service.

For the first few minutes the only change of temperature will be at the inner surface. The mean temperature of the section is scarcely changed and the E' curve has only a very small negative ordinate near its zero value at the inner surface. The approximate value of θ' at the inner surface is changed from $(\theta_1 - \theta_2)/2$ to $-(\theta_1 - \theta_2)/2$.

Thus R the radial pressure and r'/r the radial displacement are very small close to the inner surface; but T the tangential pressure and X the longitudinal pressure change from the approximate value $\frac{1}{6} \frac{\epsilon \alpha}{1-\sigma} (\theta_1 - \theta_2)$ to $-\frac{1}{6} \frac{\epsilon \alpha}{1-\sigma} (\theta_1 - \theta_2)$, that is from pressures to equal tensions.

We have seen that, for a furnace with a difference of temperature of 1000° C. between inside and outside surfaces of the walls, composed of material with a coefficient of cubical expansion $\alpha = 12 \times 10^{-6}$; a Young's modulus ϵ of 10^{11} and a Poisson's ratio σ of $\frac{1}{3}$, the value of $\frac{1}{12} \frac{\epsilon \alpha}{1-\sigma} (\theta_1 - \theta_2)$ is 1.5×10^8 dynes per square centimetre (2250 lb. per square inch), while the tensile strength of the material of the walls is of the order of 2×10^7 dynes per square centimetre (300 lb. per square inch), and that of the mortar about one-fifth of this. The sudden cooling of the interior of a furnace such as that described above will therefore subject the interior of the wall to tensile stresses in both longitudinal and tangential directions much above the tensile strength of the material, and radial cracks will be produced at the inner surface, which, since the stresses are equal in the two directions, will run indiscriminately longitudinally and tangentially.

Through the kindness of Sir Robert Hadfield, I have been able to inspect the walls of furnaces which have been in use for various periods at the Hecla Works, Sheffield, and I find that the above results are confirmed by experience.

Case 2.—A Concrete or Masonry Pillar subjected to Fire.

7. A circular pillar of radius a , initially at a uniform temperature 0° , has its outer surface suddenly raised to θ_a .

Assuming that the physical constants of the material of the pillar may be taken as uniform throughout the material, the distribution of temperature throughout the centre length of the pillar at any subsequent time will be given by the equation

$$\frac{\partial \theta}{\partial t} = \frac{k}{c\rho} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right). \quad (7.1)$$

The solution of which under the conditions laid down is

$$\theta = \theta_a - 2\theta_a \sum_{\beta} e^{-k/c\rho (\beta_n/a)^2 t} \cdot \frac{J_0(\beta_n r/a)}{\beta_n J_1(\beta_n)} \quad (7.2)$$

where $J_0(\beta)$ is the Bessel function of zero order, $J_1(\beta)$ that of the first order, and the β 's are the roots of the equation $J_0(\beta) = 0$.

* Compare Byerley, 'Fourier Series and Spherical Harmonics,' p. 229.

For a disc of the pillar initially 1 cm. thick cut perpendicular to the axis we therefore have

$$E = \int_0^r 2\pi r dr \alpha \theta = 2\pi \alpha \theta_a \left\{ \frac{r^2}{2} - 2\Sigma e^{-k/c_p (\beta_n/a)^2 t} \cdot \frac{ar \cdot J_1(\beta_n r/a)}{\beta_n^2 J_1(\beta_n)} \right\}.$$

Hence at $r = a$

$$E_a = 2\pi \alpha \theta_a \left\{ \frac{a^2}{2} - 2\Sigma e^{-k/c_p (\beta_n/a)^2 t} \cdot \left(\frac{a}{\beta_n} \right)^2 \right\}$$

and

$$E' = E - E_a \cdot \frac{r^2}{a^2} = 4\pi \alpha \theta_a \Sigma e^{-k/c_p (\beta_n/a)^2 t} \cdot \frac{r^2}{\beta_n^2} \left\{ 1 - \frac{J_1(\beta_n r/a)}{\beta_n r/a} \cdot \frac{\beta_n}{J_1(\beta_n)} \right\}. \quad (7.3)$$

Hence the stresses are given by the equations

$$R = \frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{E'}{\pi r^2} = \frac{2}{3} \cdot \frac{\epsilon}{1-\sigma} \cdot \alpha \theta_a \Sigma e^{-k/c_p (\beta_n/a)^2 t} \cdot \frac{1}{\beta_n^2} \left\{ 1 - \frac{J_1(\beta_n r/a)}{\beta_n r/a} \cdot \frac{\beta_n}{J_1(\beta_n)} \right\}, \quad (7.4)$$

$$T = \frac{1}{6} \frac{\epsilon}{1-\sigma} \left(2 \frac{dE'}{2\pi r dr} - \frac{E'}{\pi r^2} \right) = \frac{2}{3} \frac{\epsilon}{1-\sigma} \alpha \theta_a \Sigma e^{-k/c_p (\beta_n/a)^2 t} \cdot \left\{ \frac{1}{\beta_n^2} - \frac{1}{J_1(\beta_n)} \left[\frac{1}{\beta_n} J_0 \left(\frac{\beta_n}{a} r \right) - \frac{a}{r} J_1 \left(\frac{\beta_n}{a} r \right) \right] \right\}, \quad (7.5)$$

$$X = \frac{1}{6} \frac{\epsilon}{1-\sigma} 2 \frac{dE'}{2\pi r dr} = \frac{2}{3} \frac{\epsilon}{1-\sigma} \alpha \theta_a \Sigma e^{-k/c_p (\beta_n/a)^2 t} \cdot \left\{ \frac{2}{\beta_n^2} - \frac{1}{\beta_n J_1(\beta_n)} \cdot J_0 \left(\frac{\beta_n}{a} r \right) \right\}, \quad (7.6)$$

and the expansion by the equation

$$\frac{r'}{r} = \frac{1}{2} \frac{s'}{s} = \frac{2}{3} \frac{1+\sigma}{1-\sigma} \cdot \alpha \theta_a \Sigma e^{-k/c_p (\beta_n/a)^2 t} \cdot \frac{1}{\beta_n^2} \left\{ 1 - \frac{J_1(\beta_n r/a)}{\beta_n r/a} \cdot \frac{\beta_n}{J_1(\beta_n)} \right\} + \frac{1}{3} \alpha \theta_a \left\{ 1 - 4\Sigma \frac{1}{\beta_n^2} e^{-k/c_p (\beta_n/a)^2 t} \right\}, \quad (7.7)$$

the last term representing the expansion due to the mean rise of temperature.

It will be noticed that $T + R = X$, that is, the stress in the axial direction at a point is equal to the sum of the radial and tangential stresses at the point.

8. These expressions are convenient solutions of the problem when t , the time which has elapsed since the outer surface of the pillar was first heated, is sufficiently large to reduce each series to its first term, but, in general, the series converge very slowly, and many terms must be taken before an accurate numerical result is obtained.

As it is in the earlier stages of the heating, before the temperature near the middle of the pillar has commenced to rise, that the thermal stresses appear to produce their greatest effect, it is desirable to attempt to represent

the temperature distribution for small values of t by a simpler expression, which will reproduce the principal characteristics of the actual case.

We have found

$$E' = 4\pi\alpha\theta_a \Sigma e^{-k/c\rho (\beta_n/a)^2 t} \cdot \frac{r^2}{\beta_n^2} \left\{ 1 - \frac{J_1(\beta_n r/a)}{\beta_n r/a} \cdot \frac{\beta_n}{J_1(\beta_n)} \right\}.$$

Now $J_1\left(\frac{\beta_n}{a} r\right) = \frac{\beta_n}{2a} r \left\{ 1 - \frac{1}{2} \left(\frac{\beta_n r}{2a}\right)^2 + \frac{1}{12} \left(\frac{\beta_n r}{2a}\right)^4 - \dots \right\}.$

Substituting we find

$$E' = 4\pi\alpha\theta_a \Sigma e^{-k/c\rho (\beta_n/a)^2 t} \cdot \frac{r^2}{\beta_n^2} \frac{-\frac{1}{2}(\beta_n/2)^2 (a^2 - r^2)/a^2 + \frac{1}{12}(\beta_n/2)^4 (a^4 - r^4)/a^4 - \dots}{1 - \frac{1}{2}(\beta_n/2)^2 + \frac{1}{12}(\beta_n/2)^4 - \dots}, \quad (8.1)$$

and on substituting s/s_a for r/a^2

$$E' = -\alpha\theta_a s \cdot \Sigma e^{-k/c\rho \cdot (\beta_n/a)^2 t} \cdot \frac{\frac{1}{2}(s_a - s)/s_a - \frac{1}{12}(\beta_n/2)^2 (s_a^2 - s^2)/s_a^2 - \dots}{1 - \frac{1}{2}(\beta_n/2)^2 + \frac{1}{12}(\beta_n/2)^4 - \dots}.$$

The form of this expression suggests the assumption that at any time during the early stages of the heating process the value of E may be taken as approximating to $\frac{\alpha\theta_a}{n+1} \cdot s \left(\frac{s}{s_a}\right)^n$ where n has a value which decreases as t increases. Then

$$E' = -\frac{\alpha\theta_a}{n+1} s \cdot \left[1 - \left(\frac{s}{s_a}\right)^n \right], \quad (8.2)$$

$$\theta' = \frac{\theta_a}{n+1} \left\{ 1 - (n+1) \left(\frac{s}{s_a}\right)^n \right\}, \quad \bar{\theta} = \frac{\theta_a}{n+1},$$

and the values of the stresses and displacements are readily seen to be

$$R = -\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1} \left\{ 1 - \left(\frac{s}{s_a}\right)^n \right\}, \quad (8.3)$$

$$T = -\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1} \left\{ 1 - (2n+1) \left(\frac{s}{s_a}\right)^n \right\}, \quad (8.4)$$

$$X = -\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1} \left\{ 2 - 2(n+1) \left(\frac{s}{s_a}\right)^n \right\}, \quad (8.5)$$

$$\frac{r'}{r} = \frac{1}{2} \frac{s'}{s} = -\frac{1}{6} \frac{1+\sigma}{1-\sigma} \frac{\alpha\theta_a}{n+1} \left\{ 1 - (n+1) \left(\frac{s}{s_a}\right)^n \right\} + \frac{1}{3} \frac{\alpha\theta_a}{n+1}, \quad (8.6)$$

the last term giving r'/r for the mean rise of temperature $\bar{\theta}$.

Thus the mean temperature $\bar{\theta}$ increases as n decreases.

The radial stress R is zero at the surface, is a tension within, which increases in value rapidly towards its maximum value $-\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1}$ at the

centre and has nearly that value for a considerable distance from the centre if n is large.

The tangential stress T has the value $\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1} \cdot 2n$ and is a pressure at the surface, decreases to zero at the section $\left(\frac{s}{s_a}\right)^n = \frac{1}{2n+1}$ becomes a tension and has the value $-\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1}$ at the centre, attaining nearly that value at a considerable distance from the centre if n is large.

The axial stress X has the value $\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1} \cdot 2n$ and is a pressure at the surface, decreases to zero at the section $\left(\frac{s}{s_a}\right)^n = \frac{1}{n+1}$ becomes a tension and has the value $-\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1} \cdot 2$ at the centre, attaining nearly that value at a considerable distance from the centre if n is large.

The displacement term r'/r has the value $\frac{\alpha\theta_a}{n+1} \left(\frac{1}{3} + \frac{n}{6} \frac{1+\sigma}{1-\sigma}\right)$ at the surface and the displacement r' decreases to zero at the centre.

The curves of fig. 9 give the coefficients of $\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1}$ in the expressions for the stresses for the case $n = 8$. The curve for the axial stress X shows also the distribution of temperature above and below the mean temperature, and with the coefficient $\frac{1}{3} \frac{1+\sigma}{1-\sigma} \cdot \frac{\alpha\theta_a}{n+1}$ that part of the expression for the displacement term r'/r which depends on the variation of the temperature from the mean temperature.

During the early stages of the heating, the longitudinal tension X in the central portion of the section is double the radial tension R and the tangential tension T . If the pillar is not subject to other mechanical stresses, and the magnitude of these thermal stresses approaches the tensile strength of the material, cracks in planes perpendicular to the axis of the cylinder will be more frequent in the central portions than cracks in other directions.

As, however, the tensile strength of most materials is less at high than at low temperatures, it is possible that the radial tension, which persists, but with diminished magnitude in the outer and hotter layers near the surface, where the tangential and longitudinal stresses have become pressures, may still be large enough in these layers to exceed the tensile strength of the material. In any layers in which this is the case, cracks will be formed parallel to the outer surface of the cylinder.

The tangential tension T , like the radial tension R , is half the longitudinal tension X in the central portions of the section, but becomes zero, and then a

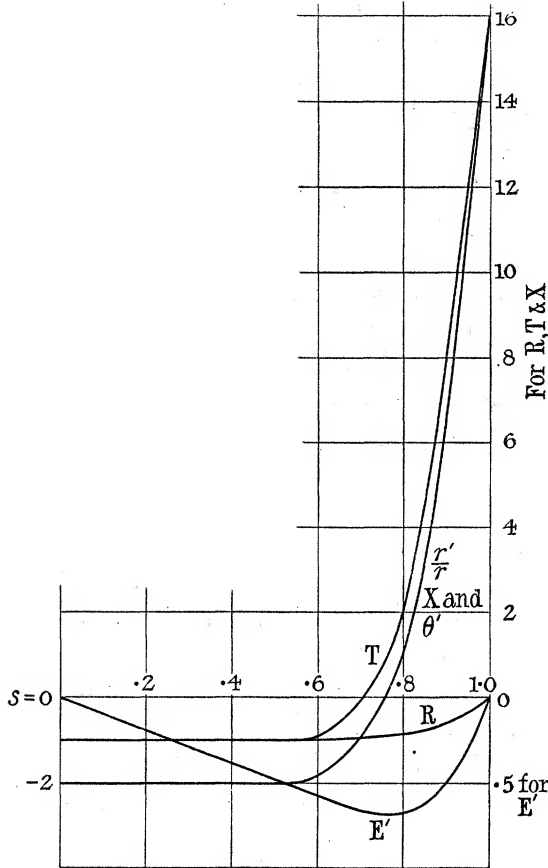


FIG. 9.—Temperature, expansion and stress curves for circular pillar, heated outside, in case $n = 8$.

pressure in the outer portions, where R is still a tension. Hence the tendency for cracks parallel to the axis to form during the heating process is less than for cracks perpendicular to the axis in the central portions, and less than for cracks parallel to the outer surface in the outer portions.

A Heated Solid Cylinder has its Surface Suddenly Cooled.

9. In this case the stresses and displacements are identical in magnitudes with those determined for the heating of the cylinder, but the sign of each is changed, a pressure becoming a tension and a tension a pressure. In fig. 9 we may consider the signs of the ordinates reversed. At the stage of the cooling shown, which we may regard as representative for the earlier stages,

the greatest tensile stress is the tangential one, T , which, except just at the surface, is slightly larger than the axial tensile stress X , and decreases rapidly in magnitude as we approach the centre. While T and X are both tensions, the radial pressure R is positive—that is a pressure. During cooling, we see, then, that there will be a slightly greater tendency for T to form cracks running parallel to the axis than for X to form cracks running round the cylinder.

There will be no tendency for R to form cracks running parallel to the outer surface of the cylinder.

In comparing the stresses produced on cooling with those due to heating, it must not be assumed in general that the contraction on cooling is identical with the dilatation on heating. As a rule, building materials expand more on first being heated than they do at any subsequent heating, and after each heating contract on cooling by a less amount than they expanded. Each heating and cooling results in a permanent increase of volume, which decreases in magnitude with each successive change.* In such cases it is therefore necessary to use the general theory of Section 2.

10. It is difficult to find experimental observations with which to compare the above deductions. The only tests of cylinders subjected to fire without mechanical loading appear to be those of W. A. Hull of the Bureau of Standards,† which were carried out on cylinders of clay, concretes, gypsums and mortars 8 inches in diameter and 16 inches long, heated in a gas furnace for $3\frac{1}{2}$ hours to 927° C. (1700° F.). These cylinders are too short to justify the neglect of the effect of the end surfaces, but the temperatures are of the order of those used in the numerical work of this paper. Mr. Hull draws no general conclusions as to the tendency of the cracks, but the illustrations show a tendency for the major cracks to run in planes perpendicular to the axis.

For longer vertical cylinders under mechanical stress due to a load carried by the cylinder, the observations of Hull‡ appear to be the only ones available for comparison. These columns were 12 or 18 inches in diameter 8 feet 9 inches long, of various materials, and were heated for 4 hours in a gas furnace to 1100° C. while under load. Both plain and reinforced concrete columns were tested.

In this case in addition to the longitudinal thermal stress X there was the

* See Day, Sosman and Hostetter, 'Amer. Jour. Sci.,' vol. 37, p. 1 (1914); and Mellor and Campbell, 'Trans. Ceramic Soc.,' vol. 15, p. 80 (1915), and references given by them.

† W. A. Hull, "Heat Insulating Properties of Materials," 'Proceedings of the American Society for Testing Materials,' Philadelphia, June, 1917.

‡ See W. A. Hull, "Fire Tests of Concrete Columns," 'Proceedings of the American Concrete Institute,' vol. 15, p. 1 (1918).

stress due to the load which, in the case of the plain concrete columns, was of the order of 800 lbs. per square inch.

According to the curves of temperature distribution given by Hull, the curves of fig. 9 correspond approximately to the state half an hour after the heating commenced. With the constants for concrete taken as $\alpha = 4 \times 10^{-5}$, $\epsilon = 1 \times 10^{11}$, $\sigma = 1/3$, the value of the coefficient $\frac{1}{6} \frac{\epsilon}{1-\sigma} \cdot \frac{\alpha \theta_a}{n+1}$ of the equations (8.4, etc.) and of the ordinates of the curves of fig. 9 is 1.2×10^8 dynes per square centimetre (1800 lbs. per square inch). Thus in the central portion of the section of the column the thermal stresses R and T are tensions of 1.2×10^8 dynes per square centimetre and X a tension of double that amount. The latter has to be diminished by the pressure due to the load and we get a resultant tension of about 2800 lbs. per square inch, which is greater than the radial and tangential tensions. There will, therefore, be a greater tendency for cracks to form perpendicular to the axis in the central parts of a column than in other directions. In the outer portion of the section the tangential stress T becomes a pressure before the radial R, and neither are affected by the loading. In these portions R is more likely to form cracks parallel to the outer surface than T is to form cracks parallel to the axis. The positions of the cracks due to R will be determined by the decrease of the tensile strength of the concrete as the temperature rises. The curves of fig. 9 show that from $s/s_a = 0.6$ to 0.8 while R decreases little the temperature increases considerably and the tensile strength therefore decreases.

Hull's observations and photographs all indicate that the materials crack along a surface about $1\frac{1}{2}$ to 2 inches within the outer surface (s/s_a , 0.6), and that the outer shell may, or may not, fall away. The existence of these cracks within columns which have been subjected to fire, but which appear on the surface uninjured, has been known for some time.*

As the tensile strength of concrete, at ordinary temperatures, is of the order 1×10^8 dynes per square centimetre (1,500 lbs. per square inch), it is seen that the thermal stresses in the heated columns, at the stage of the heating represented by fig. 9, are sufficient to account for the cracks produced.

As the heating continues the tensions in the central portion of the section at first increase, while the area subjected to them decreases. When the temperature at the centre begins to rise the stresses decrease.

Where the column is reinforced with vertical rods, with or without

* See F. J. T. Stewart, 'Report on Edison Works Fire,' Scientific and Industrial Research Department, British Fire Prevention Committee's Red Book No. 204, p. 28 (1916); and A. J. and T. D. Mylrea, 'Report on Quaker Oats Works Fire,' British Fire Prevention Committee's Red Book No. 225, p. 56 (1918).

horizontal binders, the crack produced follows the rods or binders, indicating that the tensile strength of attachment of metal to concrete is less than that of the concrete itself at the temperature attained.

So soon as the outer shell is detached from the inner portion it has to carry its share of the load without lateral support from the inside. It therefore buckles like an overloaded hollow column and eventually spalls off.

During the cooling process which, in Hull's experiments, was a gradual one, and began long before the column had wholly acquired the higher temperature, vertical cracks, starting at the outside surface and extending inwards, would be most likely to form. The tendency would be the greater the more rapid the cooling, but less than it would be if the whole column had acquired the higher temperature before cooling. The photographs of the columns, after the tests given by Hull, nearly all show these vertical cracks,* but there is no indication as to whether they were produced during the cooling or not.

So far as the experimental evidence available goes it appears to support the theoretical deductions of this paper, but it is desirable that the thermal stresses should be computed either graphically or otherwise in all cases in which the necessary observations are available, in order to determine to what extent the explanation of the behaviour of furnace walls and columns here offered is adequate.

* W. A. Hull, 'Tests of Concrete Columns,' p. 15. An important contribution to our knowledge of the effects of fire and water on loaded columns has just been made by Messrs. Ingberg, Griffin, Robinson and Wilson. See 'Bureau of Standards Technological Paper No. 184' (1921).